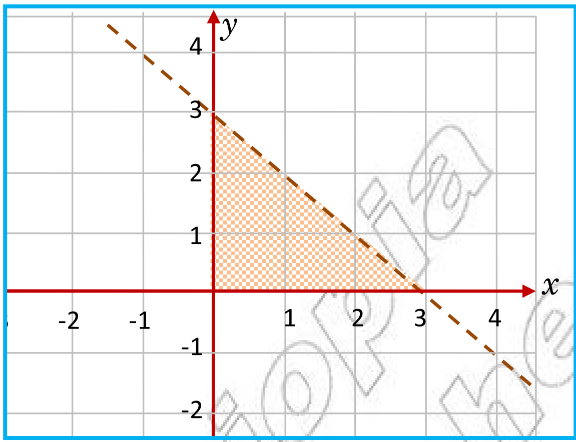
# Unit 3




# SOLVING INEQUALITIES

## Unit Outcomes:

***After completing this unit, you should be able to:***

- *know and apply methods and procedures in solving problems on inequalities involving absolute value.*
- *know and apply methods for solving systems of linear inequalities.*
- *apply different techniques for solving quadratic inequalities.*

## Main Contents

**3.1 Inequalities involving absolute value**

**3.2 Systems of linear inequalities in two variables**

**3.3 Quadratic inequalities**

***Key Terms***

***Summary***

***Review Exercises***

# INTRODUCTION

RECALL THAT OPEN STATEMENTS OF THE FORM $ax + b > 0$, $ax + b < 0$, $ax + b \le 0$ AND $ax + b \ge 0$ FOR $a \ne 0$ ARE INEQUALITIES WITH SOLUTIONS USUALLY INVOLVING INTERVALS.

IN THIS UNIT, YOU WILL STUDY METHODS OF SOLVING INEQUALITIES INVOLVING ABS SYSTEM OF LINEAR INEQUALITIES IN TWO VARIABLES AND QUADRATIC INEQUALITIES. LEARN ABOUT THE APPLICATIONS OF THESE METHODS IN SOLVING PRACTICAL PROB INEQUALITIES.

## 3.1 INEQUALITIES INVOLVING ABSOLUTE VALUE

THE METHODS FREQUENTLY USED FOR DESCRIBING SETS ARE THE COMPLETE LISTING METHOD, THE PARTIAL LISTING METHOD AND THE SET-BUILDER METHOD. SETS OF REAL NUMBERS C BE DESCRIBED BY USING THE SET-BUILDER METHOD OR INTERVALS (*sets of real numbers between any two given real numbers*).

**Notation:** FOR REAL NUMBERS $a$ AND $b$ WHERE $a < b$,

- ✓ $(a, b)$ IS AN OPEN INTERVAL;
- ✓ $(a, b]$ AND $[a, b)$ ARE HALF CLOSED OR HALF OPEN INTERVALS; AND
- ✓ $[a, b]$ IS A CLOSED INTERVAL.

FOR EXAMPLE, (5, 9) IS THE SET OF REAL NUMBERS BETWEEN 5 AND 9 AND [5, 9] IS THE SET NUMBERS BETWEEN 5 AND 9 INCLUDING 5 AND 9.

THAT IS, $(5, 9) = \{x: 5 < x < 9 \text{ AND } x \in \mathbb{R}\}$

$[5, 9] = \{x : 5 \le x \le 9 \text{ AND } x \in \mathbb{R}\}$

IN GENERAL, IF $a$ AND $b$ ARE FIXED REAL NUMBERS WITH $a < b$, THEN

$[a, b] = \{x: a \le x \le b \text{ AND } x \in \mathbb{R}\}$ $\quad (a, b) = \{x: a < x < b \text{ AND } x \in \mathbb{R}\}$

$(-\infty, a] = \{x: x \le a \text{ AND } x \in \mathbb{R}\}$ $\quad (a, \infty) = \{x: x > a \text{ AND } x \in \mathbb{R}\}$

**Note:** THE SYMBOL "$\infty$" IS USED TO MEAN ***positive infinity*** AND "$-\infty$" IS USED TO MEAN ***negative infinity***.

GRAPHICALLY, $[a, b]$ IS

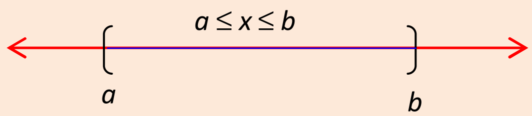



Figure 3.1

GRAPHICALLY, $(a, b)$ IS

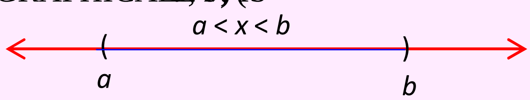



Figure 3.2

GRAPHICALLY, $(-\infty, a]$ IS

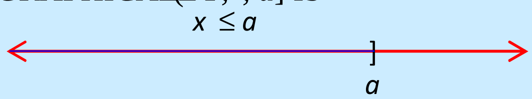



Figure 3.3

GRAPHICALLY, $(a, \infty)$ IS

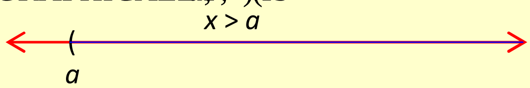



Figure 3.4

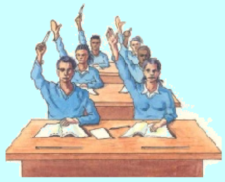
INTERVALS ARE COMMONLY USED TO EXPRESS THE SOLUTION SETS OF INEQUALITIES. FO US FIND THE SOLUTION SET OF $2x + 4 \leq 3x - 5$.

$2x + 4 \leq 3x - 5$ IS EQUIVALENT TO $2x - 3x \leq -5 - 4$ WHICH IS $-x \leq -9$.

MULTIPLYING BOTH SIDES BY $-1$ GIVES $x \geq 9$. (REMEMBER THAT, WHEN YOU MULTIPLY OR DIVIDE BY A NEGATIVE NUMBER, THE INEQUALITY SIGN IS CHANGED.)

SO, THE SOLUTION SET IS $[9, \infty)$.

## ACTIVITY 3.1



**1** DISCUSS THE 3-METHODS OF DESCRIBING SETS: THE *complete listing method*, THE *partial listing method* AND THE *set-builder method*.

**2** GIVE EXAMPLES FOR EACH OF THE METHODS USED FOR DESCRIBING A SET.

**3** DESCRIBE EACH OF THE FOLLOWING SETS USING ANY ONE OF THE METHODS:

- **A** THE SET OF NUMBERS $-3, -2, 1, 0, 2, 3$.
- **B** THE SET OF ALL NEGATIVE MULTIPLES OF 2.
- **C** THE SET OF NATURAL NUMBERS GREATER THAN 0.6 AND LESS THAN 5.

**4** DESCRIBE EACH OF THE FOLLOWING SETS USING SET-BUILDER METHOD:

- **A** $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
- **B** $\{0, 3, 6, 9, \ldots\}$
- **C** $[-3, 5)$
- **D** $[2, \infty)$

**5** WRITE EACH OF THE FOLLOWING USING INTERVALS:

- **A** $\{x : x \in \mathbb{R} \setminus \{0\}\}$
- **B** $\{x : -1 \leq x \leq 2 \text{ AND } x \in \mathbb{R}\}$
- **C** $\{x : 0.2 < x \leq 0.8 \text{ AND } x \in \mathbb{R}\}$
- **D** $\{x : x \in \mathbb{R} \text{ AND } x \neq -1\}$

**6** FIND ALL VALUES OF $x$ SATISFYING THE FOLLOWING INEQUALITIES:

- **A** $2x - 1 < 7$
- **B** $4 \leq 1 - x < 5$

LOOK AT THE NUMBER LINE GIVEN BELOW.

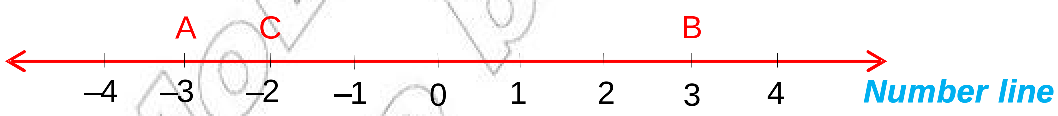



Figure 3.5

WHAT ARE THE COORDINATES OF POINTS A AND B ON THE NUMBER LINE?

WHAT IS THE DISTANCE OF POINT A FROM THE ORIGIN? WHAT ABOUT B?

THE NUMBER THAT SHOWS ONLY THE DISTANCE FROM THE POINT CORRESPONDING TO ZERO (NOT THE DIRECTION) IS CALLED THE ***absolute value***. FOR EXAMPLE, THE POINT C (WITH COORDINATE –2) IS 2 UNITS FROM THE POINT CORRESPONDING TO ZERO. THIS IS DENOTED BY $|-2| = 2$.

ON THE NUMBER LINE, $|x|$ IS THE DISTANCE BETWEEN THE POINT CORRESPONDING TO NUMBER $x$ AND THE POINT CORRESPONDING TO ZERO, REGARDLESS OF WHETHER THE POINT IS TO THE LEFT OR RIGHT OF THE POINT CORRESPONDING TO ZERO AS SHOWN IN FIGURE 3.6 BELOW.

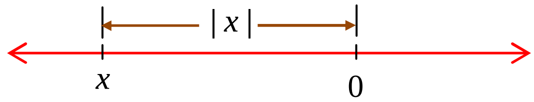



Figure 3.6

### Definition 3.1

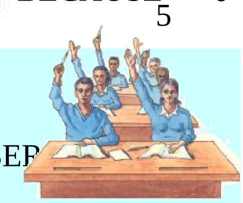
If $x$ is a real number, then the absolute value of $x$, denoted by $|x|$, is defined by

$$|x| = \begin{cases} x, & \text{IF } x \geq 0 \\ -x, & \text{IF } x < 0 \end{cases}$$

**EXAMPLE 1**

**A** $|25| = 25$ BECAUSE $25 > 0$ **B** $\left|-\frac{4}{5}\right| = -\left(-\frac{4}{5}\right) = \frac{4}{5}$ BECAUSE $-\frac{4}{5} < 0$

## ACTIVITY 3.2



**1** WHY IS IT ALWAYS TRUE THAT FOR ANY REAL NUMBER $x$, $|x| \geq 0$?

**2** EVALUATE EACH OF THE FOLLOWING EXPRESSIONS:

**A** $|-3|$ **B** $|0|$ **C** $|-\sqrt{5}|$

**D** $|-3|\ |-2|$ **E** $|1-\sqrt{2}|$ **F** $|\sqrt{3}-\sqrt{5}|$

**3** IF $x = -2$ AND $y = 3$, THEN EVALUATE EACH OF THE FOLLOWING:

**A** $|6x + y|$ **B** $|6x| + |y|$ **C** $|2x - 3y|$

**4** VERIFY EACH OF THE FOLLOWING USING EXAMPLES:

**A** $|x - y| = |y - x|$ **B** $|2x-3y| = |3y-2x|$ **C** $\sqrt{x^2} = |x|$

**D** $|x|\ |y| = |xy|$ **E** $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$

GEOMETRICALLY, THE EQUATION $|x| = 5$ MEANS THAT THE POINT WITH COORDINATE $x$ IS 5 UNITS AWAY FROM THE POINT CORRESPONDING TO ZERO, ON THE NUMBER LINE. OBVIOUSLY, THE NUMBER LINE CONTAINS TWO POINTS THAT ARE 5 UNITS FROM THE POINT CORRESPONDING TO ZERO; ONE TO THE LEFT AND THE OTHER TO THE RIGHT. SO, $|x| = 5$ HAS TWO SOLUTIONS, $x = 5$ AND $x = -5$.

**Theorem 3.1 Solutions of the equation $|x| = a$**

For any real number $a$, the equation $|x| = a$ has

**I** two solutions $x = a$ and $x = -a$, if $a > 0$;

**II** one solution, $x = 0$, if $a = 0$; and

**III** no solution, if $a < 0$.

**EXAMPLE 2** SOLVE EACH OF THE FOLLOWING ABSOLUTE VALUE EQUATIONS:

**A** $|3x + 5| = 2$ **B** $\left|\frac{2}{3}x + 1\right| = 0$ **C** $|2x - 1| = -3$

**SOLUTION:**

**A** $|3x + 5| = 2$ IS EQUIVALENT TO $3x + 5 = -2$ OR $3x + 5 = 2$

$\Rightarrow 3x + 5 - 5 = -2 - 5$ OR $3x + 5 - 5 = 2 - 5$

$\Rightarrow 3x = -7$ OR $3x = -3$

$\Rightarrow x = -\frac{7}{3}$ OR $x = -1$

THEREFORE, $x = -\frac{7}{3}$ AND $x = -1$ ARE THE TWO SOLUTIONS.

**B** WE KNOW THAT $|x| = 0$ IF AND ONLY IF $x = 0$. THEREFORE, $\left|\frac{2}{3}x + 1\right| = 0$ IS EQUIVALENT TO $\frac{2}{3}x + 1 = 0$. HENCE, $\frac{2}{3}x = -1$

$\Rightarrow x = -\frac{3}{2}$ IS THE SOLUTION.

**C** SINCE $|x| \geq 0$ FOR ALL $x \in \mathbb{R}$, THE GIVEN EQUATION $|2x - 1| = -3$ HAS NO SOLUTION.

AS DISCUSSED ABOVE, $|x| = 4$ MEANS $x = -4$ OR $x = 4$. HENCE, ON THE NUMBER LINE, THE POINT CORRESPONDING TO $x$ IS 4 UNITS AWAY FROM THE POINT CORRESPONDING TO 0. WE SEE TH $|x| \leq 4$, THE DISTANCE BETWEEN THE POINT CORRESPONDING TO $x$ AND THE POINT CORRESPONDING TO 0 IS LESS THAN 4 OR EQUAL TO 4. IT FOLLOWS THAT $|x| \leq 4$ IS EQUIVALENT TO $-4 \leq x \leq 4$.

WE HAVE THE FOLLOWING GENERALIZATION.

**Theorem 3.2 Solution of $|x| < a$ and $|x| \leq a$**

For any real number $a > 0$,

**I** the solution of the inequality $|x| < a$ is $-a < x < a$.

**II** the solution of the inequality $|x| \leq a$ is $-a \leq x \leq a$.

**EXAMPLE 3** SOLVE EACH OF THE FOLLOWING ABSOLUTE VALUE INEQUALITIES:

**A** $|2x - 5| < 3$ **B** $|3 - 5x| \le 1$

**SOLUTION:**

**A** $|2x - 5| < 3$ IS EQUIVALENT TO $-3 < 2x - 5 < 3$,

$\Rightarrow$ $-3 < 2x - 5$ AND $2x - 5 < 3$

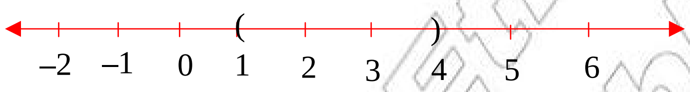
$\Rightarrow$ $-3 + 5 < 2x - 5 + 5$ AND $2x - 5 + 5 < 3 + 5$

$\Rightarrow$ $2 < 2x$ AND $2x < 8$

$\Rightarrow$ $1 < x$ AND $x < 4$ THAT IS, $1 < x < 4$

THEREFORE, THE SOLUTION SET IS $\{x : 1 < x < 4\} = (1, 4)$

WE CAN REPRESENT THE SOLUTION SET ON THE NUMBER LINE AS FOLLOWS:




**B** $|3 - 5x| \le 1$ IS EQUIVALENT TO $-1 \le 3 - 5x \le 1$

$\Rightarrow$ $-1 \le 3 - 5x$ AND $3 - 5x \le 1$

$\Rightarrow$ $-1 - 3 \le 3 - 3 - 5x$ AND $3 - 3 - 5x \le 1 - 3$

$\Rightarrow$ $-4 \le -5x$ AND $-5x \le -2$

$\Rightarrow$ $5x \le 4$ AND $2 \le 5x$

$\Rightarrow$ $x \le \frac{4}{5}$ AND $x \ge \frac{2}{5}$ THAT IS, $\frac{2}{5} \le x \le \frac{4}{5}$

THEREFORE, THE SOLUTION SET IS $\left\{x : \frac{2}{5} \le x \le \frac{4}{5}\right\} = \left[\frac{2}{5}, \frac{4}{5}\right]$

**Note:** IN $|x| < a$, IF $a < 0$ THE INEQUALITY $|x| < a$ HAS NO SOLUTION.

**Theorem 3.3** **Solution of $|x| > a$ and $|x| \ge a$**

For any real number $a$, if $a > 0$, then

**I** the solution of the inequality $|x| > a$ is $x < -a$ or $x > a$.

**II** the solution of the inequality $|x| \ge a$ is $x \le -a$ or $x \ge a$.

**EXAMPLE 4** SOLVE EACH OF THE FOLLOWING INEQUALITIES:

**A** $|5 + 2x| > 6$ **B** $\left|\frac{3}{5} - 2x\right| \ge 1$ **C** $|3 - x| > -2$

**SOLUTION:** ACCORDING TO THEOREM 3.3,

**A** $|5 + 2x| > 6$ IMPLIES $5 + 2x < -6$ OR $5 + 2x > 6$

$\Rightarrow 5 - 5 + 2x < -6 - 5$ OR $5 - 5 + 2x > 6 - 5$

$\Rightarrow 2x < -11$ OR $2x > 1$

$\Rightarrow x < \frac{-11}{2}$ OR $x > \frac{1}{2}$

THEREFORE, THE SOLUTION SET IS $\left\{x : x < -\frac{11}{2} \text{ OR } x > \frac{1}{2}\right\}$.

(TRY TO REPRESENT THIS SOLUTION ON THE NUMBER LINE)

**B** $\left|\frac{3}{5} - 2x\right| \geq 1$ IMPLIES $\frac{3}{5} - 2x \leq -1$ OR $\frac{3}{5} - 2x \geq 1$

HENCE, $\frac{3}{5} - 2x \leq -1$ OR $\frac{3}{5} - 2x \geq 1$ GIVES $\frac{3}{5} - \frac{3}{5} - 2x \leq -1 - \frac{3}{5}$ OR $\frac{3}{5} - \frac{3}{5} - 2x \geq 1 - \frac{3}{5}$

$\Rightarrow -2x \leq \frac{-8}{5}$ OR $-2x \geq \frac{2}{5}$

$\Rightarrow \frac{8}{5} \leq 2x$ OR $-\frac{2}{5} \geq 2x$

$\Rightarrow x \geq \frac{4}{5}$ OR $x \leq -\frac{1}{5}$

THEREFORE, THE SOLUTION SET IS $\left\{x : x \leq -\frac{1}{5} \text{ OR } x \geq \frac{4}{5}\right\}$.

**C** BY DEFINITION, $|3 - x| = |x - 3| \geq 0$. SO, $|3 - x| > -2$ IS TRUE FOR ALL REAL NUMBERS $x$

THEREFORE, THE SOLUTION SET IS $\mathbb{R}$.

## Group Work 3.1

**1** GIVEN THAT $0 < a < b$, EXPRESS THE FOLLOWING WITHOUT ABSOLUTE VALUE.

**A** $|a - b|$ **B** $|ab - a|$ **C** $\left|\frac{b}{a}\right|$

**2** FOR ANY REAL NUMBER $a$, SHOW THAT

**A** $a \leq |a|$

**Hint:** IF $a \geq 0$, THEN $|a| = a$. SO, $a \leq |a|$.
IF $a < 0$, THEN $|a| > 0$. COMPARE $a$ AND $|a|$.

**B** $-|a| \leq a \leq |a|$

**3** FOR ANY REAL NUMBERS $x$ AND $y$, SHOW THAT

**A** $|x + y| \le |x| + |y|$

**Hint:** START FROM $|x + y|^2 = (x + y)^2$ AND EXPAND. THEN USE 2B ABOVE.

**B** $|x - y| \ge |x| - |y|$

**4** SOLVE EACH OF THE FOLLOWING

**A** $\frac{3x - 1}{2} + x \le 7 + \frac{1}{2}x$ **B** $|-2| \ge 8 - |4x + 6|$

**C** $\left|\frac{1}{4}x - 2\right| > 1$ **D** $|2x - 1| < x + 3$

## Exercise 3.1

**1** SIMPLIFY AND WRITE EACH OF THE FOLLOWING USING INTERVALS:

**A** $\{x : x \in \mathbb{R} \text{ AND } x \ne -2\}$ **B** $\{x : -1 \le x - 2 \le 2\}$

**C** $\{x : x + 3 > 2\}$ **D** $\{x : 5x - 9 \le 9\}$

**E** $\{x : 2x + 3 \ge -5x\}$ **F** $\{x : 2x - 1 < x < 3\}$

**2** SOLVE EACH OF THE FOLLOWING INEQUALITIES:

**A** $2x - 5 \ge 3x$ **B** $3x + 1 < \frac{8x - 3}{2}$ **C** $\frac{1}{4}t + 2 > 3(5 - t)$

**3** A NUMBER $y$ IS 15 LARGER THAN A POSITIVE NUMBER $x$. IF THEIR SUM IS NOT MORE THAN 85, WHAT ARE THE POSSIBLE VALUES OF SUCH NUMBER $y$?

**4** IF $x = -\frac{2}{3}$ AND $y = \frac{1}{5}$, THEN EVALUATE THE FOLLOWING:

**A** $|6x| + |5y|$ **B** $|3x| - |10y|$ **C** $|3x - 10y|$ **D** $\left|\frac{3x - 2y}{x + y}\right|$

**5** SOLVE EACH OF THE FOLLOWING ABSOLUTE VALUE EQUATIONS:

**A** $|3x + 6| = 7$ **B** $|5x - 3| = 9$ **C** $|x - 6| = -6$

**D** $|7 - 2x| = 0$ **E** $|6 - 3x| + 5 = 14$ **F** $\left|\frac{3}{4}x + \frac{1}{8}\right| = \frac{1}{2}$

**6** SOLVE EACH OF THE FOLLOWING ABSOLUTE VALUE INEQUALITIES AND EXPRESS THEIR SOLUTION SETS IN INTERVALS:

**A** $|3 - 5x| \le 1$ **B** $|5x| - 2 < 8$ **C** $\left|\frac{2}{3}x - \frac{1}{9}\right| \ge \frac{1}{3}$

**D** $|6 - 2x| + 3 > 8$ **E** $|3x + 5| \le 0$ **F** $|x - 1| > -2$

**7** FOR ANY REAL NUMBERS $a$, $b$ AND $c$ SUCH THAT $a \ne 0$ AND $c \ge 0$, SOLVE EACH OF THE FOLLOWING INEQUALITIES:

**A** $|ax + b| < c$ **B** $|ax + b| \le c$ **C** $|ax + b| > c$ **D** $|ax + b| \ge c$

## 3.2 SYSTEMS OF LINEAR INEQUALITIES IN TWO VARIABLES

RECALL THAT A FIRST DEGREE (LINEAR) EQUATION IN TWO VARIABLES $x$ AND $y$ HAS THE FORM

$$ax + by = c$$

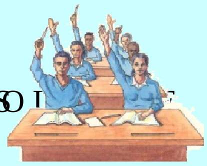
WHERE $a$ AND $b$ BOTH ARE NOT 0.

WHEN TWO OR MORE LINEAR EQUATIONS INVOLVE THE SAME VARIABLES, THEY ARE CA ***of linear equations***. AN ORDERED PAIR THAT SATISFIES ALL THE EQUATIONS OF A SYSTEM IS CALLED A ***solution of the system***. FOR INSTANCE

$$\begin{cases} 2x - y = 7 \\ x + 5y = -2 \end{cases}$$

IS A SYSTEM OF TWO LINEAR EQUATIONS. WHAT IS ITS SOLUTION?

### ACTIVITY 3.3



**1** WHAT CAN YOU SAY ABOUT THE SOLUTION SET OF TWO LINEAR EQUATIONS IF THEIR GRAPHS DO NOT INTERSECT?

**2** FIND THE SOLUTIONS OF EACH OF THE FOLLOWING SYSTEMS OF EQUATIONS GRAPHICALLY:

**A** $\begin{cases} x - y = -2 \\ x + y = 6 \end{cases}$ **B** $\begin{cases} x + y = 2 \\ 2x + 2y = 8 \end{cases}$ **C** $\begin{cases} x + 2y = 4 \\ 2x + 4y = 8 \end{cases}$

**3** FIND THREE DIFFERENT ORDERED PAIRS WHICH BELONG TO R WHERE

R = $\{(x, y): y \le x + 1\}$.

**4** DRAW THE GRAPH OF R GIVEN IN QUESTION 3 ABOVE.

**5** DRAW THE GRAPHS OF EACH OF THE FOLLOWING RELATIONS:

**A** R = $\{(x, y): x \ge y$ AND $y \ge x - 1\}$ **B** R = $\{(x, y): y \le x + 1$ AND $y \ge 1 - x\}$.

**6** SOLVE EACH OF THE FOLLOWING SYSTEMS OF INEQUALITIES AND WRITE YOUR ANSWER IN INTERVAL NOTATION:

**A** $\begin{cases} x \ge -1 \\ x \le 3 \\ y \ge 0 \end{cases}$ **B** $\begin{cases} x - y < 3 \\ x \ge 2 \end{cases}$

A SYSTEM OF TWO LINEAR EQUATIONS IN TWO VARIABLES OFTEN INVOLVES A PAIR OF IN THE PLANE. THE SOLUTION SET OF SUCH A SYSTEM OF EQUATIONS CAN BE DETERMI GRAPH AND IS THE SET OF ALL ORDERED PAIRS OF COORDINATES OF POINTS WHICH LIE

**EXAMPLE 1** FIND THE SOLUTION SET OF THE SYSTEM OF EQUATIONS $\begin{cases} x - y = 3 \\ x + 2y = 0 \end{cases}$

**SOLUTION:** FIRST DRAW THE GRAPHS OF $x - y = 3$ AND $x + 2y = 0$ AS SHOWN BELOW.

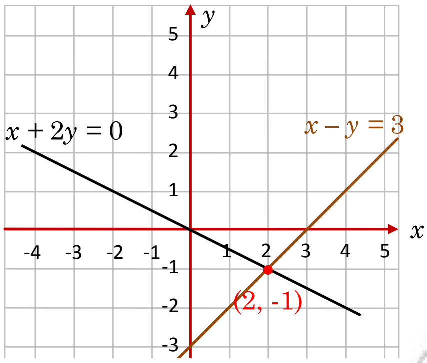



Figure 3.7

THE TWO LINES INTERSECT AT (2, –1).

THEREFORE, THE SOLUTION SET OF THE SYSTEM IS {(2, –1)}.

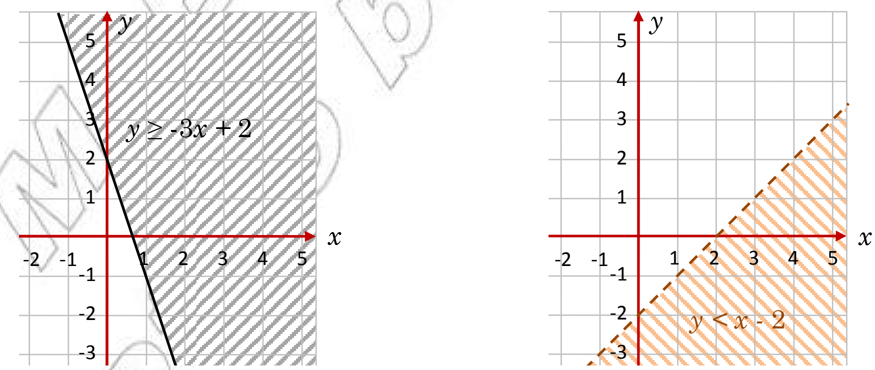
IN A SYSTEM OF EQUATIONS, IF "=" IS REPLACED BY "<", "≤", ">" OR "≥", THE SYSTEM BECOMES A SYSTEM OF LINEAR INEQUALITIES.

**EXAMPLE 2** FIND THE SOLUTION OF THE FOLLOWING SYSTEM OF INEQUALITIES GRAPHICALLY:

$$\begin{cases} y \geq -3x + 2 \\ y < x - 2 \end{cases}$$

**SOLUTION:** FIRST DRAW THE GRAPH OF ONE OF THE BOUNDARY LINES, $y = -3x + 2$, CORRESPONDING TO THE FIRST INEQUALITY.

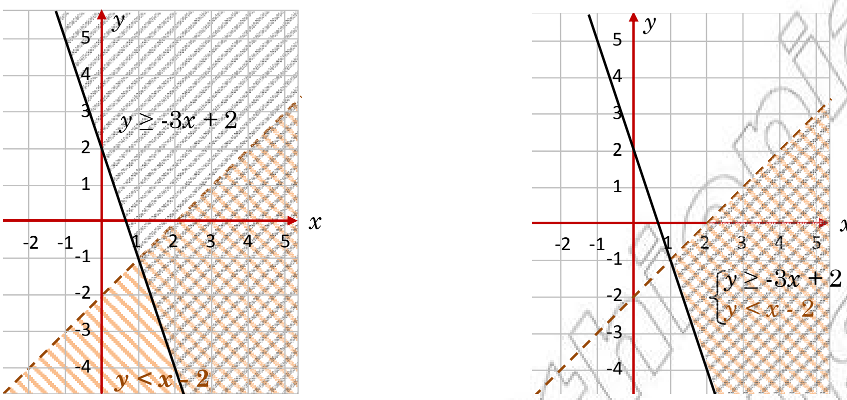
THE GRAPH OF $y \geq -3x + 2$ CONSISTS OF POINTS ON OR ABOVE THE LINE $y = -3x + 2$ AS SHOWN IN FIGURE 3.8A. THIS IS OBTAINED BY TAKING A TEST POINT, SAY (2, 0), AND CHECKING THAT $0 \geq -3(2) + 2 = -4$ IS TRUE. NEXT, DRAW THE GRAPH OF THE OTHER BOUNDARY LINE, $y = x - 2$, CORRESPONDING TO THE SECOND INEQUALITY. THE GRAPH OF $y < x - 2$ CONSISTS OF POINTS BELOW THE LINE $y = x - 2$. POINTS ON THE LINE ARE EXCLUDED AS SHOWN IN FIGURE 3.8B.




A B

Figure 3.8

THESE GRAPHS HAVE BEEN DRAWN USING DIFFERENT COORDINATE SYSTEMS IN ORDER TO SHOW THEM SEPARATELY. NOW, DRAW THEM USING THE SAME COORDINATE SYSTEM. THE REGION IN THE COORDINATE SYSTEM MARKED WITH BOTH TYPES OF SHADING IS THE SOLUTION SET OF THE SYSTEM, AS SHOWN IN FIGURE 3.9B.




A B

Figure 3.9

THE SOLUTION SET OF $\begin{cases} y \geq -3x+2 \\ y < x-2 \end{cases}$ IS SHOWN BY THE CROSS-SHADED REGION IN THE DIAGRAM.

SOLVING $\begin{cases} y = -3x+2 \\ y = x-2 \end{cases}$, WE GET $-3x + 2 = x - 2$

THEREFORE, $x = 1$ AND $y = -1$

SO, $x > 1$, $-3x + 2 \leq y < x - 2$

HENCE, THE SOLUTION SET OF THE SYSTEM IS EXPRESSED AS

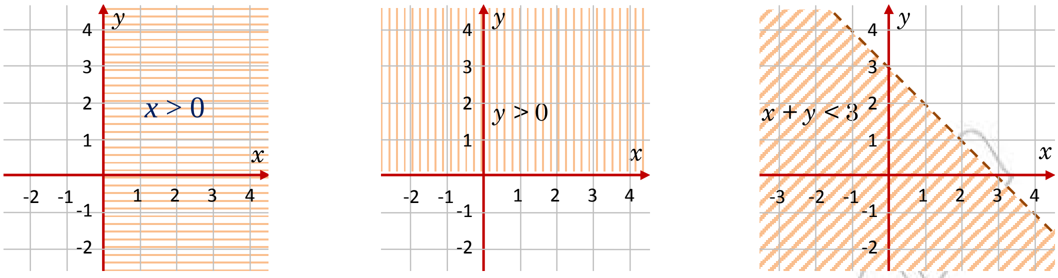
$$\{(x, y): -3x+2 \leq y < x-2 \text{ AND } 1 < x < \infty\}$$

**EXAMPLE 3** FIND THE SOLUTION OF EACH OF THE FOLLOWING SYSTEMS OF LINEAR INEQUALITIES, GRAPHICALLY:

**A** $\begin{cases} x+y<3 \\ x \geq 0 \\ y \geq 0 \end{cases}$ **B** $\begin{cases} y+x>0 \\ y-x \leq 1 \\ x \leq 2 \end{cases}$

**SOLUTION:**

**A** HERE, OUR OBJECTIVE IS TO DETERMINE THE SET OF POINTS WHOSE COORDINATES ($x$, $y$) SATISFY ALL THREE OF THESE CONDITIONS. TO DO SO, LET US DRAW EACH BOUNDARY LINE AS SHOWN BELOW. THE POINTS SATISFYING THE CONDITION $x \geq 0$ ARE THOSE LYING TO THE RIGHT OF THE $y$-AXIS AS SHOWN IN FIGURE 3.10A.




A B C

Figure 3.10

THE POINTS $(x, y)$ WITH $y > 0$ ARE THE POINTS THAT LIE ABOVE THE $x$-AXIS AS SHOWN IN FIGURE 3.10B. THE POINTS $(x, y)$ WITH $x + y < 3$ IS THE SET OF POINTS LYING BELOW THE LINE $x + y = 3$. POINTS ON THE LINE ARE EXCLUDED.

NOW, DRAW THE GRAPH OF THE THREE INEQUALITIES $x \geq 0$, $y \geq 0$ AND $x + y < 3$, USING THE SAME COORDINATE SYSTEM, TAKING ONLY THE INTERSECTION OF THE THREE REGI

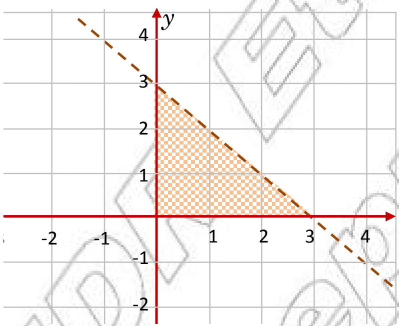
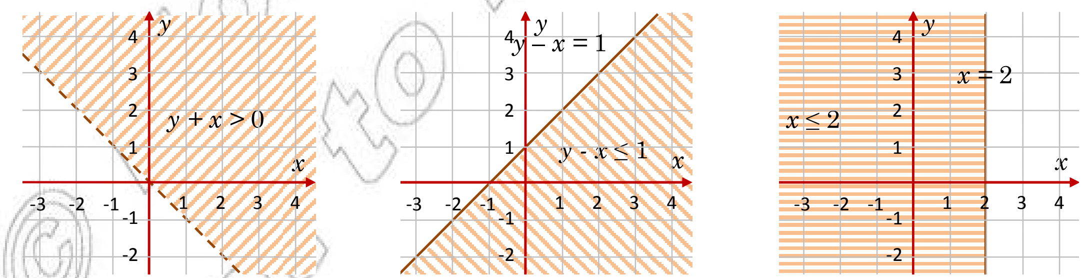


Figure 3.11

THE POINTS SATISFYING THE SYSTEM OF INEQUALITIES ARE THE POINTS THAT SATISFY ALL THE THREE INEQUALITIES. THE CORRESPONDING REGION IS THE TRIANGULAR REGION FIGURE 3.11. THAT IS, THE SET OF $(x, y)$ SUCH THAT $x \in [0, 3)$ AND $y \in [0, 3 - x)$

**b** FIRST, DRAW THE GRAPH OF THE BOUNDARY LINE $y + x = 0$ (OR $y = -x$) FOR THE FIRST INEQUALITY. THE GRAPH OF $y + x > 0$ CONSISTS OF POINTS ABOVE THE LINE.

POINTS ON THE LINE ARE EXCLUDED AS SHOWN IN FIGURE 3.12A



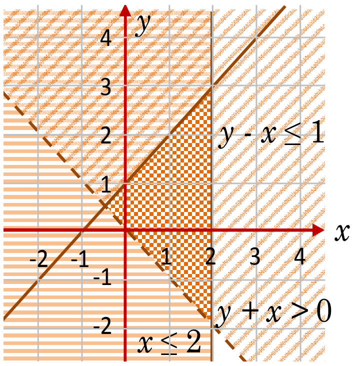

A B C

Figure 3.12

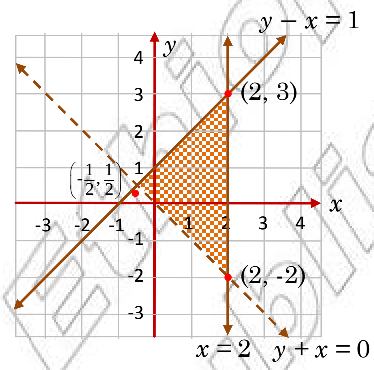
NEXT, DRAW THE GRAPH OF THE BOUNDARY LINE $y - x = 1$ FOR THE SECOND INEQUALITY. THE GRAPH OF $y - x \leq 1$ CONSISTS OF POINTS ON AND BELOW THE LINE $y - x = 1$ AS SHOWN IN FIGURE 3.12B.

FINALLY, DRAW THE GRAPH OF THE BOUNDARY LINE $x = 2$ FOR THE THIRD INEQUALITY. THE POINTS $(x, y)$ SATISFYING THE CONDITION $x \leq 2$ ARE THOSE LYING ON AND TO THE LEFT OF TH LINE $x = 2$ AS SHOWN IN FIGURE 3.12C.

NOW, DRAW THE GRAPH OF THE THREE INEQUALITIES USING THE SAME COO SYSTEM AS SHOWN IN FIGURE 3.13A.




A

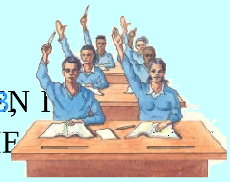



B

Figure 3.13

BECAUSE THERE ARE INFINITE SOLUTIONS TO THE SYSTEM, THEY CANNOT BE LISTED. BUT THE GRAPH IS EASY TO DESCRIBE. THE SOLUTION IS THE TRIANGULAR R VERTICES $\left(-\frac{1}{2}, \frac{1}{2}\right)$, $(2, 3)$ AND $(2, -2)$, EXCEPT THOSE POINTS ON THE LINE $y + x = 0$, AS SHOWN IN FIGURE 3.13B.

## ACTIVITY 3.4



**1** BY OBSERVING THE GRAPH OF THE INEQUALITY GIVEN IN FIGURE 3.13B, NAME AT LEAST 10 ORDERED PAIRS THAT SATISFY THE

**2** IF R = $\{(x, y): y + x > 0, y - x \leq 1$ AND $x \leq 2\}$, WHAT IS THE DOMAIN AND RANGE OF R?

WE SHALL NOW CONSIDER AN EXAMPLE INVOLVING AN APPLICATION OF A SYSTEM INEQUALITIES.

**EXAMPLE 4** A FURNITURE COMPANY MAKES TABLES AND CHAIRS. TO PRODUCE A TABLE IT REQUIRES 2 HRS ON MACHINE A, AND 4 HRS ON MACHINE B. TO PRODUCE A IT REQUIRES 3 HRS ON MACHINE A AND 2 HRS ON MACHINE B. MACHINE A OPERATE AT MOST 12 HRS A DAY AND MACHINE B CAN OPERATE AT MOST DAY. IF THE COMPANY MAKES A PROFIT OF BIRR 12 ON A TABLE AND BIRR CHAIR, HOW MANY OF EACH SHOULD BE PRODUCED TO MAXIMIZE ITS PROFI

**SOLUTION:** LET $x$ BE THE NUMBER OF TABLES TO BE PRODUCED AND $y$ BE THE NUMBER OF CHAIRS TO BE PRODUCED.

THEN, IF A TABLE IS PRODUCED IN 2 HRS ON MACHINE A, $x$ TABLES REQUIRE $2x$ HRS. SIMILARLY, $y$ CHAIRS REQUIRE $3y$ HRS ON MACHINE A. ON MACHINE B, $x$ TABLES REQUIRE $4x$ HRS AND $y$ CHAIRS REQUIRE $2y$ HRS. SINCE MACHINES A AND B CAN OPERATE AT MOST 12 HRS AND 16 HRS, RESPECTIVELY, YOU HAVE THE FOLLOWING SYSTEM OF LINEAR INEQUALIT

FROM MACHINE A: $2x + 3y \leq 12$

FROM MACHINE B: $4x + 2y \leq 16$

ALSO, $x \geq 0$ AND $y \geq 0$ SINCE $x$ AND $y$ ARE NUMBERS OF TABLES AND CHAIRS.

HENCE, YOU OBTAIN A SYSTEM OF LINEAR INEQUALITIES GIVEN AS FOLLOWS:

$$\begin{cases} 2x + 3y \leq 12 \\ 4x + 2y \leq 16 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

SINCE THE INEQUALITIES INVOLVED IN THE SYSTEM ARE ALL LINEAR, THE BOUNI GRAPH OF THE SYSTEM ARE STRAIGHT LINES. THE REGION CONTAINING THE SOI SYSTEM IS THE QUADRILATERAL SHOWN BELOW.

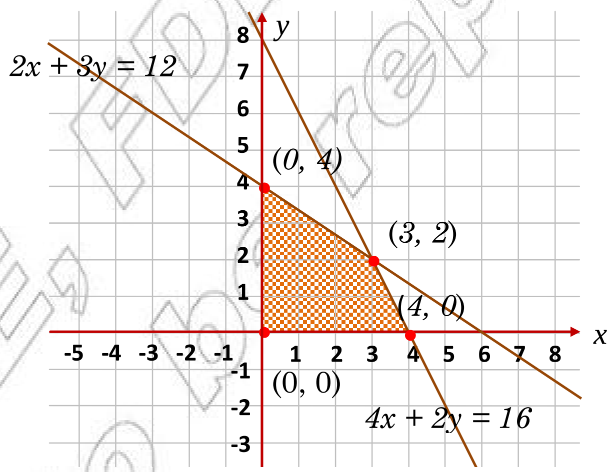



Figure 3.14

THE PROFIT MADE IS BIRR 12 ON A TABLE, SO BIRR $12x$ ON $x$ TABLES AND BIRR 10 ON A CHAIR, SO BIRR $10y$ ON $y$ CHAIRS. THE PROFIT FUNCTION P IS GIVEN BY $P = 12x + 10y$.

THE VALUES OF $x$ AND $y$ WHICH MAXIMIZE OR MINIMIZE THE PROFIT FUNCTION ON SUCH SYSTEM ARE USUALLY FOUND AT VERTICES OF THE SOLUTION REGION.

HENCE, FROM THE GRAPH, YOU HAVE THE COORDINATES OF EACH VERTEX AS FIGURE 3.14.

THE PROFIT: P = 12$x$ + 10$y$ AT EACH VERTEX IS FOUND TO BE:

AT (0, 0), P = 12 (0) + 10 (0) = 0

AT (0, 4), P = 12 (0) + 10 (4) = 40

AT (3, 2), P = 12 (3) + 10 (2) = 56

AT (4, 0), P = 12 (4) + 10 (0) = 48

THEREFORE, THE PROFIT IS MAXIMUM AT THE VERTEX (3, 2), SO THE COMPANY PRODUCE 3 TABLES AND 2 CHAIRS PER DAY TO GET THE MAXIMUM PROFIT OF BIRR

## Group Work 3.2

**1** FIND THE SOLUTIONS OF EACH OF THE FOLLOWING LINEAR INEQUALITIES GRAPHICALLY:

**A** $\begin{cases} y + x \geq 0 \\ y - x \geq 0 \\ y \leq 3 \end{cases}$ **B** $\begin{cases} x + y < 1 \\ 2x - y > -1 \\ y - 3x \geq -3 \end{cases}$

**2** LET R = {($x$, $y$): $y \geq x$, $y \geq -x$ AND $y \leq 3$} AND

$r$ = {($x$, $y$): $x + y < 1$, $2x - y > -1$ AND $y - 3x \geq -3$}

USING QUESTION 1 ABOVE, FIND THE DOMAIN AND RANGE OF THE RELATIONS $R$ AND $r$.

## Exercise 3.2

**1** DRAW THE GRAPHS OF EACH OF THE FOLLOWING RELATIONS:

**A** R = {($x$, $y$) : $x - y \geq 1$ AND $2x + y < 3$}

**B** R = {($x$, $y$) : $x \leq y - 1$ AND $y + 2x > 2$}

**C** R = {($x$, $y$) : $x > y$ ; $x > 0$ AND $y + x < 1$}

**D** R = {($x$, $y$) : $x + y \geq 0$ ; $y \geq 0$ AND $x + y < 1$}

**2** SOLVE EACH OF THE FOLLOWING SYSTEM OF INEQUALITIES GRAPHICALLY:

**A** $\begin{cases} y \leq 2x + 3 \\ y - x \geq 0 \\ y > 0 \end{cases}$ **B** $\begin{cases} 3x + y < 5 \\ x > 0 \\ x + y < 6 \end{cases}$ **C** $\begin{cases} y \leq 1 - x \\ y > x + 2 \\ y > 0 \end{cases}$

**D** $\begin{cases} x \geq -1 \\ y \leq 2 \\ y \geq x - 1 \end{cases}$ **E** $\begin{cases} 0 \leq x \leq 1 \\ 0 \leq y \leq 1 \end{cases}$ **F** $\begin{cases} x > 0 \\ y > 0 \\ x + y < 4 \end{cases}$

**3** DESCRIBE EACH OF THE FOLLOWING SHADED REGIONS BY A SYSTEM OF LINEAR INEQUALITIES:

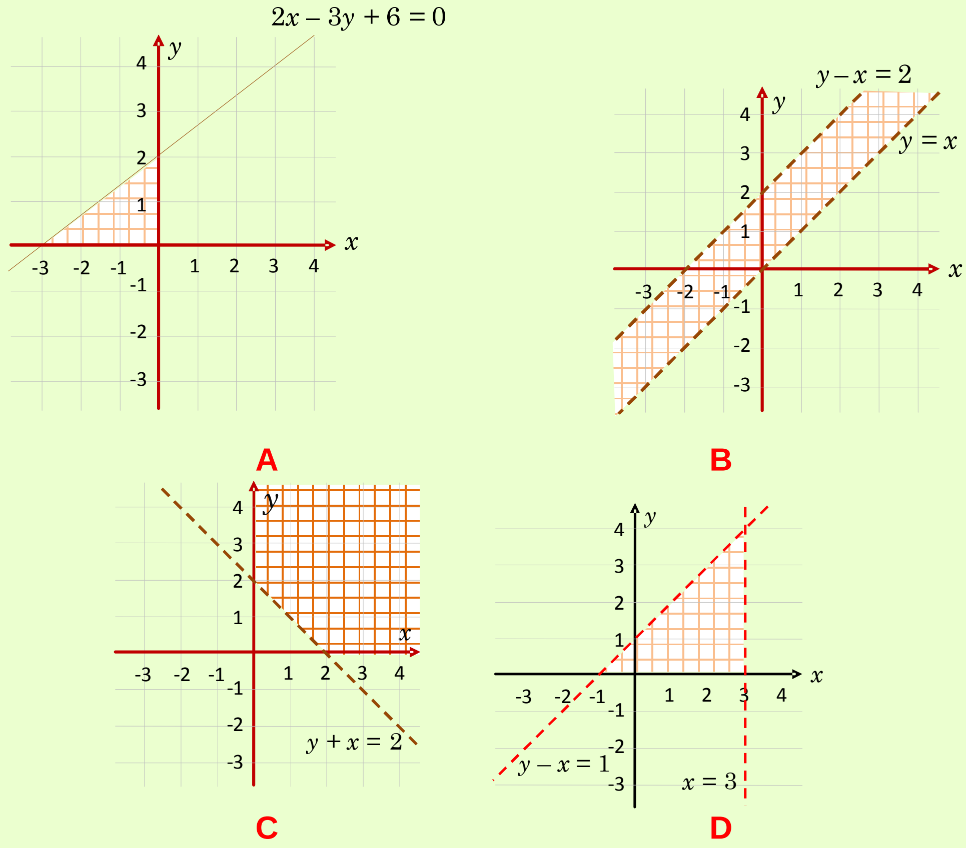



Figure 3.15

**4** GIVE A PAIR OF LINEAR INEQUALITIES THAT DESCRIBE ALL THE POINTS IN THE FIRST QUADRANT.

**5** GIVE A SYSTEM OF LINEAR INEQUALITIES WHOSE SOLUTION IS ALL THE POINTS INSIDE A RECTANGLE.

**6** SUPPOSE THE SUM OF TWO POSITIVE NUMBERS $x$ AND $y$ IS LESS THAN 10 AND GREATER THAN 5. SHOW ALL POSSIBLE VALUES FOR $x$ AND $y$ GRAPHICALLY.

**7** SUPPOSE A SHOE FACTORY PRODUCES BOTH LOW-GRADE AND HIGH-GRADE SHOES. THE FACTORY PRODUCES AT LEAST TWICE AS MANY LOW-GRADE AS HIGH-GRADE MAXIMUM POSSIBLE PRODUCTION IS 500 PAIRS OF SHOES. A DEALER CALLS FOR DE AT LEAST 100 HIGH-GRADE PAIRS OF SHOES PER DAY. SUPPOSE THE OPERATION PROFIT OF BIRR 2.00 PER A PAIR OF SHOES ON HIGH-GRADE SHOES AND BIRR 1.00 P OF SHOES ON LOW-GRADE SHOES. HOW MANY PAIRS OF SHOES OF EACH TYPE S PRODUCED FOR MAXIMUM PROFIT?

**Hint:** LET $x$ DENOTE THE NUMBER OF HIGH-GRADE SHOES.
LET $y$ DENOTE THE NUMBER OF LOW-GRADE SHOES.

## 3.3 QUADRATIC INEQUALITIES

IN UNIT 2 OF GRADE 9 MATHEMATICS, YOU HAVE LEARNT HOW TO SOLVE QUADRATIC EQUATIONS. (RECALL THAT EQUATIONS OF THE FORM $ax^2 + bx + c = 0$, $a \neq 0$ ARE QUADRATIC EQUATIONS.)

*Can similar methods be used to solve quadratic inequalities?*

**Definition 3.2**

An ***inequality*** that can be reduced to any one of the following forms:

$ax^2 + bx + c \leq 0$ or $ax^2 + bx + c < 0$,

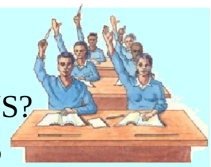
$ax^2 + bx + c \geq 0$ or $ax^2 + bx + c > 0$,

where $a$, $b$ and $c$ are constants and $a \neq 0$, is called a ***quadratic inequality***.

FOR EXAMPLE, $x^2 - 3x + 2 < 0$, $x^2 + 1 \geq 0$, $x^2 + x \leq 0$ AND $x^2 - 4 > 0$ ARE ALL QUADRATIC INEQUALITIES.

THE FOLLOWING ACTIVITY WILL HELP YOU TO RECALL WHAT YOU HAVE LEARNED A EQUATIONS IN GRADE 9.

## ACTIVITY 3.5



**1** WHICH OF THE FOLLOWING ARE QUADRATIC EQUATIONS?

- **A** $x - 2 = x^2 + 2x$ **B** $x^2 - 2x = x^2 + 3x + 6$
- **C** $2(x - 4) - (x - 2) = (x + 2)(x - 4)$ **D** $x^3 - 3 = 1 + 4x + x^2$
- **E** $(x - 1)(x + 2) \geq 0$ **F** $x(x - 1)(x + 1) = 0$.

**2** WHICH OF THE FOLLOWING ARE QUADRATIC INEQUALITIES?

- **A** $2x^2 \leq 5x + x^2 - 3$ **B** $2x^2 > 2x + x^2 + 8$
- **C** $x(1 - x) \leq (x + 2)(1 - x)$ **D** $3x^2 + 5x + 6 > 0$
- **E** $5 - 2(x^2 + x) < 6x - 2x^2$ **F** $(x - 2)(x + 1) \geq 2 - 2x$
- **G** $-1 > (x^2 + 1)(x + 2)$.

**3** IF THE PRODUCT OF TWO REAL NUMBERS IS ZERO, WHAT CAN YOU SAY ABOUT THE TWO NUMBERS?

**4** FACTORIZE EACH OF THE FOLLOWING IF POSSIBLE:

- **A** $x^2 + 6x$ **B** $35x - 28x^2$ **C** $\frac{1}{16} - 25x^2$ **D** $4x^2 + 7x + 3$
- **E** $x^2 - x + 3$ **F** $x^2 + 2x - 3$ **G** $3x^2 - 11x - 4$ **H** $x^2 + 4x + 4$.

**5** GIVEN A QUADRATIC EQUATION $ax^2 + bx + c = 0$,

- **A** WHAT IS ITS DISCRIMINANT?
- **B** STATE WHAT MUST BE TRUE ABOUT THE DISCRIMINANT IF THE EQUATION HAS ONE REAL ROOT, TWO DISTINCT REAL ROOTS, AND NO REAL ROOT.

## 3.3.1 Solving Quadratic Inequalities Using Product Properties

SUPPOSE YOU WANT TO SOLVE THE QUADRATIC INEQUALITY

$$(x - 2)(x + 3) > 0.$$

CHECK THAT $x = 3$ MAKES THE STATEMENT TRUE WHILE $x = 1$ MAKES IT FALSE. HOW DO YOU FIND THE SOLUTION SET OF THE GIVEN INEQUALITY? OBSERVE THAT THE LEFT HAND SIDE O IS THE PRODUCT OF $x - 2$ AND $x + 3$. THE PRODUCT OF TWO REAL NUMBERS IS POSITIVE, IF AN ONLY IF EITHER BOTH ARE POSITIVE OR BOTH ARE NEGATIVE. THIS FACT CAN BE USE GIVEN INEQUALITY.

### Product properties:

**1** $m.n > 0$, if and only if

**I** $m > 0$ and $n > 0$ or **II** $m < 0$ and $n < 0$.

**2** $m.n < 0$, if and only if

**I** $m > 0$ and $n < 0$ or **II** $m < 0$ and $n > 0$.

**EXAMPLE 1** SOLVE EACH OF THE FOLLOWING INEQUALITIES:

**A** $(x + 1)(x - 3) > 0$ **B** $3x^2 - 2x \geq 0$

**C** $-2x^2 + 9x + 5 < 0$ **D** $x^2 - x - 2 \leq 0$

**SOLUTION:**

**A** BY PRODUCT PROPERTY 1, $(x + 1)(x - 3)$ IS POSITIVE IF EITHER BOTH THE FACTORS ARE POSITIVE OR BOTH ARE NEGATIVE.

NOW, CONSIDER CASE BY CASE AS FOLLOWS:

**Case i** WHEN BOTH THE FACTORS ARE POSITIVE

$x + 1 > 0$ AND $x - 3 > 0$

$x > -1$ AND $x > 3$

THE INTERSECTION OF $x > -1$ AND $x > 3$ IS $x > 3$. THIS CAN BE ILLUSTRATED ON THE NUMBER LINE AS SHOWN IN FIGURE 3.16 BELOW.

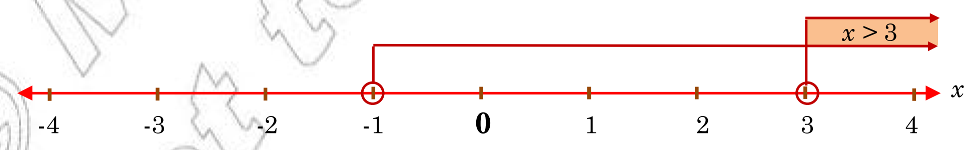



Figure 3.16

THE SOLUTION SET FOR THIS FIRST CASE IS $S_1 = \{x : x > 3\} = (3, \infty)$.

**Case ii** WHEN BOTH THE FACTORS ARE NEGATIVE

$x + 1 < 0$ AND $x - 3 < 0$

$x < -1$ AND $x < 3$

THE INTERSECTION OF $x < -1$ AND $x < 3$ IS $x < -1$.

THIS CAN BE ILLUSTRATED ON THE NUMBER LINE AS SHOWN BELOW IN FIGURE 3.17

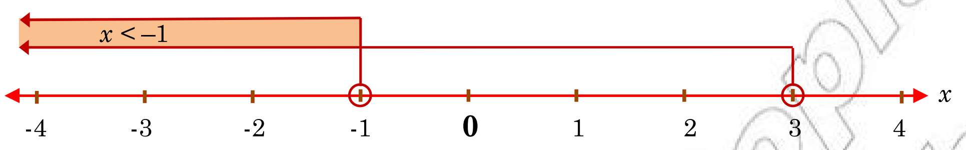



Figure 3.17

THE SOLUTION SET FOR THIS SECOND CASE IS $S_2 = \{x: x < -1\} = (-\infty, -1)$.

THEREFORE, THE SOLUTION SET OF $(x + 1)(x - 3) > 0$ IS:

$S_1 \cup S_2 = \{x: x < -1 \text{ OR } x > 3\} = (-\infty, -1) \cup (3, \infty)$

**B** FIRST, FACTORIZE $3x^2 - 2x$ AS $x(3x - 2)$

SO, $3x^2 - 2x \geq 0$ MEANS $x(3x - 2) \geq 0$ EQUIVALENTLY,

I $x \geq 0$ AND $3x - 2 \geq 0$ OR

II $x \leq 0$ AND $3x - 2 \leq 0$

**Case i** WHEN $x \geq 0$ AND $3x - 2 \geq 0$

$x \geq 0$ AND $x \geq \frac{2}{3}$

THE INTERSECTION OF $x \geq 0$ AND $x \geq \frac{2}{3}$ IS $x \geq \frac{2}{3}$. GRAPHICALLY,

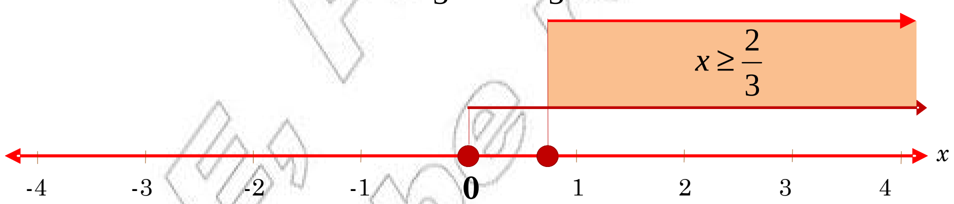



Figure 3.18

SO, $S_1 = \{x : x \geq \frac{2}{3}\} = [\frac{2}{3}, \infty)$

**Case ii** WHEN $x \leq 0$ AND $3x - 2 \leq 0$ THAT IS $x \leq 0$ AND $x \leq \frac{2}{3}$

THE INTERSECTION OF $x \leq 0$ AND $x \leq \frac{2}{3}$ IS $x \leq 0$. GRAPHICALLY,

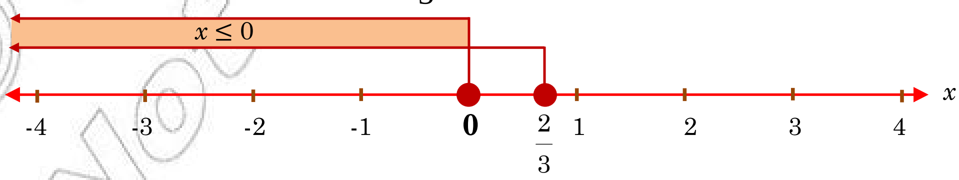



Figure 3.19

SO, $S_2 = \{x: x \le 0\} = (-\infty, 0]$

THEREFORE, THE SOLUTION SET FOR $2x^2 \ge 3x$ IS

$$S_1 \cup S_2 = \{x: x \le 0 \text{ OR } x \ge \frac{2}{3}\} = (-\infty, 0] \cup [\frac{2}{3}, \infty)$$

**c** $-2x^2 + 9x + 5 = (-2x - 1)(x - 5) < 0$

BY PRODUCT PROPERTY 2, $(-2x - 1)(x - 5)$ IS NEGATIVE IF ONE OF THE FACTORS IS NEGATIVE AND THE OTHER IS POSITIVE.

AS BEFORE, CONSIDER CASE BY CASE AS FOLLOWS:

**Case i** WHEN $-2x - 1 > 0$ AND $x - 5 < 0$

$$x < -\frac{1}{2} \text{ AND } x < 5$$

THE INTERSECTION OF $x < -\frac{1}{2}$ AND $x < 5$ IS $x < -\frac{1}{2}$. GRAPHICALLY,

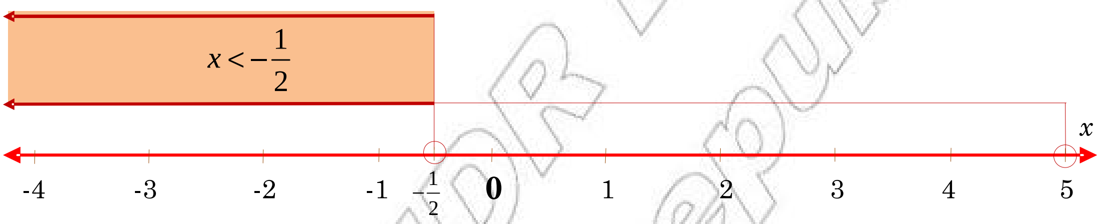



Figure 3.20

SO, $S_1 = \{x: x < -\frac{1}{2}\} = (-\infty, -\frac{1}{2})$

**Case ii** WHEN $-2x - 1 < 0$ AND $x - 5 > 0$

$$x > -\frac{1}{2} \text{ AND } x > 5$$

THE INTERSECTION OF $x > 5$ AND $x > -\frac{1}{2}$ IS $x > 5$. GRAPHICALLY,

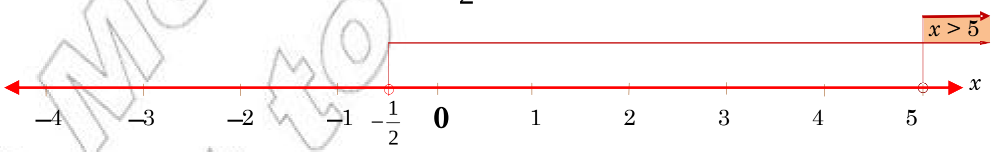



Figure 3.21

SO, $S_2 = \{x: x > 5\} = (5, \infty)$

THEREFORE, THE SOLUTION SET FOR $(-2x - 1)(x - 5) < 0$ IS

$$S_1 \cup S_2 = \{x: x < -\frac{1}{2} \text{ OR } x > 5\} = (-\infty, -\frac{1}{2}) \cup (5, \infty)$$

**D** $x^2 - x - 2 = (x + 1)(x - 2)$

SO, $x^2 - x - 2 \le 0$ MEANS $(x + 1)(x - 2) \le 0$

BY PRODUCT PROPERTY 2, $(x + 1)(x - 2)$ IS NEGATIVE IF ONE OF THE FACTORS IS NEGATIVE AND THE OTHER IS POSITIVE. TO SOLVE $(x + 1)(x - 2) \le 0$, CONSIDER CASE BY CASE AS FOLLOWS:

**Case i** $x + 1 \ge 0$ AND $x - 2 \le 0$

$x \ge -1$ AND $x \le 2$

THE INTERSECTION OF $x \ge -1$ AND $x \le 2$ IS $-1 \le x \le 2$. GRAPHICALLY,

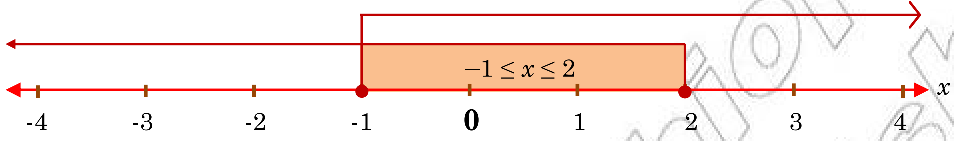



Figure 3.22

SO, $S_1 = \{x: -1 \le x \le 2\} = [-1, 2]$

**Case ii** $x + 1 \le 0$ AND $x - 2 \ge 0$

$x \le -1$ AND $x \ge 2$

THERE IS NO INTERSECTION OF $x \le -1$ AND $x \ge 2$. GRAPHICALLY,

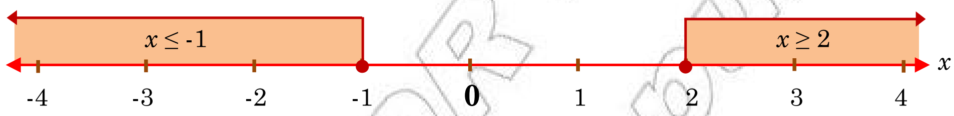



Figure 3.23

SO, $S_2 = \varnothing$

THEREFORE, THE SOLUTION SET FOR $x^2 - x - 2 \le 0$ IS

$S_1 \cup S_2 = \{x: -1 \le x \le 2\} \cup \varnothing = \{x: -1 \le x \le 2\} = [-1, 2]$

## Exercise 3.3

**1** SOLVE EACH OF THE FOLLOWING INEQUALITIES USING PRODUCT PROPERTIES.

**A** $x(x + 5) > 0$ **B** $(x - 1)^2 \le 0$

**C** $(4 + x)(x - 4) > 0$ **D** $(5x - 3)(x + 7) < 0$

**E** $(1 + x)(3 - 2x) \ge 0$ **F** $(5 - x)(1 - \frac{1}{3}x) \le 0$

**2** FACTORIZE AND SOLVE EACH OF THE FOLLOWING INEQUALITIES USING PRODUCT PROPERTIES:

**A** $x^2 + 5x + 4 < 0$ **B** $x^2 - 4 > 0$ **C** $x^2 + 5x + 6 \ge 0$

**D** $x^2 - 2x + 1 \le 0$ **E** $3x^2 + 4x + 1 \ge 0$ **F** $2x^2 - 7x + 3 < 0$

**G** $25x^2 - \frac{1}{16} < 0$ **H** $x^2 + 4x + 4 > 0$.

3 a FIND THE SOLUTION SET OF THE INEQUALITY $x^2 < 25$.

b WHY IS $\{x \mid x < 5\}$ NOT THE SOLUTION SET OF $x^2 < 25$?

4 IF $x < y$, DOES IT FOLLOW THAT $x^2 < y^2$?

5 IF A BALL IS THROWN UPWARD FROM GROUND LEVEL WITH INITIAL VELOCITY OF 24 M/S, ITS HEIGHT H IN METRES AFTER T SECONDS IS GIVEN BY $h(t) = 24t - 16t^2$. WHEN WILL THE BALL BE AT A HEIGHT OF MORE THAN 8 METRES?

## 3.3.2 Solving Quadratic Inequalities Using the Sign Chart Method

SUPPOSE YOU NEED TO SOLVE THE QUADRATIC INEQUALITY

$$x^2 + 3x - 4 < 0.$$

CONSIDER HOW THE SIGN OF $x^2 + 3x - 4$ CHANGES AS YOU VARY THE VALUES OF THE UNKNOW AS $x$ IS MOVED ALONG THE NUMBER LINE, THE QUANTITY $x^2 + 3x - 4$ IS SOMETIMES POSITIVE, SOMETIMES ZERO, AND SOMETIMES NEGATIVE. TO SOLVE THE INEQUALITY, YOU MUS VALUES OF $x$ FOR WHICH $x^2 + 3x - 4$ IS NEGATIVE. INTERVALS WHERE $x^2 + 3x - 4$ IS POSITIVE ARE SEPARATED FROM INTERVALS WHERE IT IS NEGATIVE BY VALUES OF $x$ FOR WHICH IT IS ZERO. TO LOCATE THESE VALUES, SOLVE THE EQUATION $x^2 + 3x - 4 = 0$.

FACTORIZE $x^2 + 3x - 4$ AND FIND THE TWO ROOTS (–4 AND 1). DIVIDE THE NUMBER LINE I THREE OPEN INTERVALS. THE EXPRESSION $x^2 + 3x - 4$ WILL HAVE THE SAME SIGN IN EACH OF THESE INTERVALS: $(-\infty, -4)$, $(-4, 1)$ AND $(1, \infty)$.

THE "SIGN CHART" METHOD ALLOWS YOU TO FIND THE SIGN OF $x^2 + 3x - 4$ IN EACH INTERVAL.

**Step 1** FACTORIZE: $x^2 + 3x - 4 = (x + 4)(x - 1)$

**Step 2** DRAW A SIGN CHART, NOTING THE SIGN OF EACH FACTOR AND THE WHOLE EXPRESSION AS SHOWN BELOW.

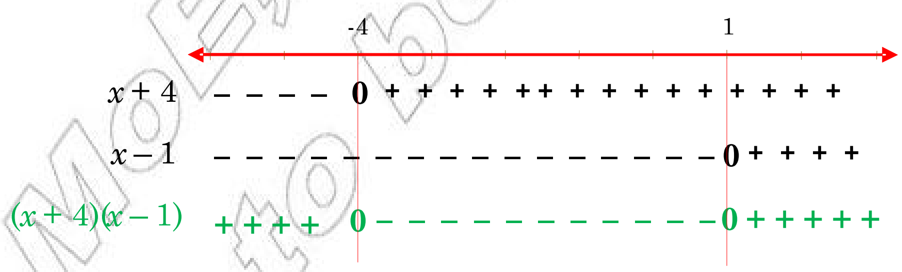



Figure 3.24

**Step 3** READ THE SOLUTION FROM THE LAST LINE OF THE SIGN CHART.

$$x^2 + 3x - 4 < 0 \text{ FOR } x \in (-4, 1)$$

THEREFORE, THE SOLUTION SET IS THE INTERVAL (–4, 1)

**EXAMPLE 2** SOLVE EACH OF THE FOLLOWING INEQUALITIES USING THE SIGN CHART METHOD:

**A** $6 + x - x^2 \leq 0$  **B** $2x^2 + 3x - 2 \geq 0$.

**SOLUTION:**

**A** FACTORIZE $6 + x - x^2$ SO THAT $6 + x - x^2 = (x + 2)(3 - x) \leq 0$.

WE MAY IDENTIFY THE SIGNS OF $x + 2$ AND $3 - x$ AS FOLLOWS.

$x + 2 < 0$ FOR EACH $x < -2$, $x + 2 = 0$ AT $x = -2$ AND $x + 2 > 0$ FOR EACH $x > -2$.

SIMILARLY, $3 - x > 0$ FOR EACH $x < 3$, $3 - x = 0$ AT $x = 3$ AND $3 - x < 0$ FOR EACH $x > 3$.

THEREFORE, THE ABOVE RESULTS ARE SHOWN IN THE SIGN CHART GIVEN BELOW IN FIGURE 3.25.

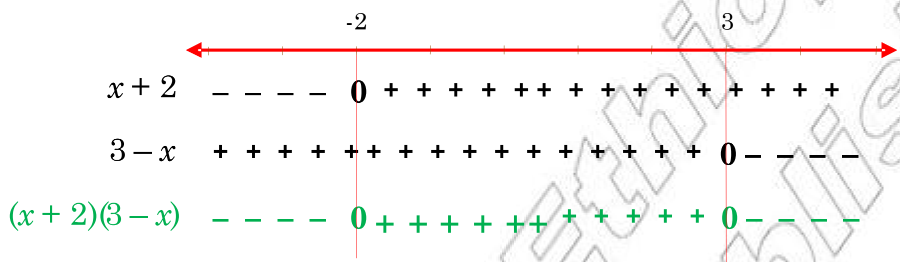



Figure 3.25

FROM THE SIGN CHART, YOU CAN IMMEDIATELY READ THE FOLLOWING:

**I** THE SOLUTION SET OF $(x + 2)(3 - x) < 0$ IS $\{x: x < -2 \text{ OR } x > 3\} = (-\infty, -2) \cup (3, \infty)$.

**II** THE SOLUTION SET OF $(x + 2)(3 - x) > 0$ IS $\{x: -2 < x < 3\} = (-2, 3)$.

**III** THE SOLUTION SET OF $(x + 2)(3 - x) = 0$ IS $\{-2, 3\}$.

**IV** THE SOLUTION SET OF $(x + 2)(3 - x) \leq 0$ IS $(-\infty, -2] \cup [3, \infty)$

THEREFORE, THE SOLUTION SET OF $6 + x - x^2 \leq 0$ IS $(-\infty, -2] \cup [3, \infty)$.

**B** $2x^2 + 3x - 2 = (2x - 1)(x + 2) \geq 0$.

$2x - 1 < 0$ FOR EACH $x < \frac{1}{2}$, $2x - 1 = 0$ AT $x = \frac{1}{2}$, AND $2x - 1 > 0$ FOR EACH $x > \frac{1}{2}$.

SIMILARLY, $x + 2 < 0$ FOR EACH $x < -2$, $x + 2 = 0$ AT $x = -2$ AND $x + 2 > 0$ FOR EACH $x > -2$.

THE ABOVE RESULTS ARE SHOWN IN THE SIGN CHART GIVEN BELOW:

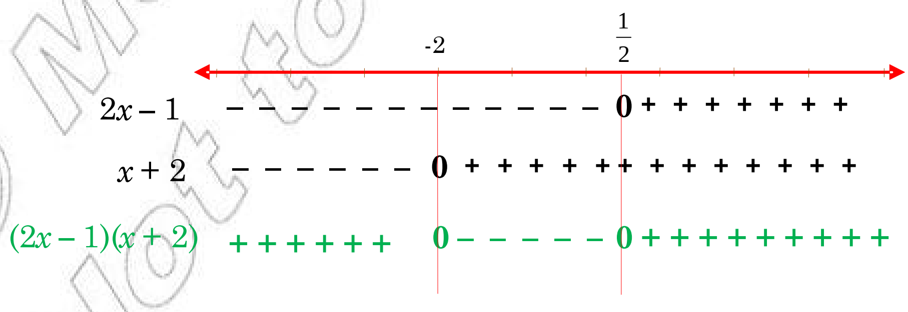



Figure 3.26

FROM THE SIGN CHART, YOU CAN CONCLUDE THAT

$(2x - 1)(x + 2) \geq 0$ FOR EACH $x \in (-\infty, -2] \cup \left[\frac{1}{2}, \infty\right)$ AND

$(2x - 1)(x + 2) < 0$ FOR EACH $x \in \left(-2, \frac{1}{2}\right)$.

THEREFORE, THE SOLUTION SET OF $2x^2 + 3x - 2 \geq 0$ IS $(-\infty, -2] \cup \left[\frac{1}{2}, \infty\right)$

**EXAMPLE 3** FOR WHAT VALUE(S) OF $k$ DOES THE QUADRATIC EQUATION $kx^2 - 2x + k = 0$ HAS

**I** ONLY ONE REAL ROOT? **II** TWO DISTINCT REAL ROOTS?

**III** NO REAL ROOTS?

**SOLUTION:** THE QUADRATIC EQUATION $kx^2 - 2x + k = 0$ IS EQUIVALENT TO THE QUADRATIC EQUATION $ax^2 + bx + c = 0$ WITH $a = k$, $b = -2$ AND $c = k$

THE GIVEN QUADRATIC EQUATION HAS

**I** ONE REAL ROOT WHEN $b^2 - 4ac = 0$

SO, $(-2)^2 - 4(k)(k) = 0$

$4 - 4k^2 = 0$ EQUIVALENTLY $(2 - 2k)(2 + 2k) = 0$

$2 - 2k = 0$ OR $2 + 2k = 0$

$k = 1$ OR $k = -1$

THEREFORE, $kx^2 - 2x + k = 0$ HAS ONLY ONE REAL ROOT IF EITHER $k = 1$ OR $k = -1$.

**II** TWO DISTINCT REAL ROOTS WHEN $b^2 - 4ac > 0$

IT FOLLOWS THAT, $4 - 4k^2 > 0$

$(2 - 2k)(2 + 2k) > 0 \Rightarrow 4(1 - k)(1 + k) > 0$

NOW, USE THE SIGN CHART SHOWN BELOW:

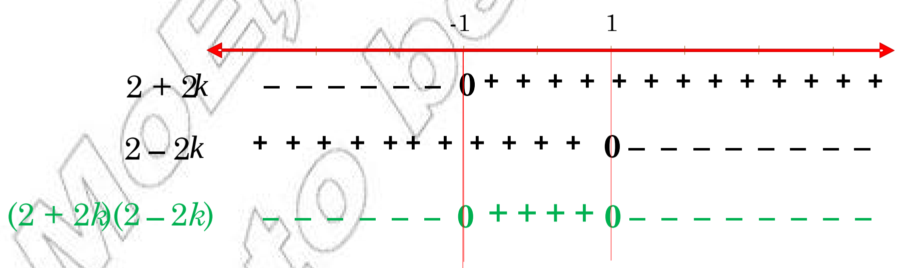



Figure 3.27

THEREFORE, FOR EACH $k \in (-1, 1)$, THE GIVEN QUADRATIC EQUATION HAS TWO DISTINCT REAL ROOTS.

**III** $kx^2 - 2x + k = 0$ HAS NO REAL ROOT FOR EACH $k \in (-\infty, -1) \cup (1, \infty)$ WHERE $B^2 - 4AC < 0$

*What do you do if $ax^2 + bx + c$, $a \neq 0$ is not factorizable into linear factors?*

THAT IS, THERE ARE NO REAL NUMBERS $x_1$ AND $x_2$ SUCH THAT $ax^2 + bx + c = a(x - x_1)(x - x_2)$.

IN THIS CASE, EITHER $ax^2 + bx + c > 0$ FOR ALL VALUES OF $x$ OR $ax^2 + bx + c < 0$ FOR ALL VALUES OF $x$.

AS A RESULT, THE SOLUTION SET OF $ax^2 + bx + c > 0$ OR $ax^2 + bx + c \geq 0$ IS EITHER $(-\infty, \infty)$ OR { }.

TAKE A TEST POINT AND SUBSTITUTE, IN ORDER TO DECIDE WHICH IS THE CASE.

**EXAMPLE 4** SOLVE EACH OF THE FOLLOWING QUADRATIC INEQUALITIES:

**A** $x^2 - 2x + 5 \geq 0$ **B** $-3x^2 + x - 1 \geq 0$.

**SOLUTION:**

**A** FOR $x^2 - 2x + 5 \geq 0$

$a = 1$, $b = -2$, $c = 5$ AND $b^2 - 4ac = (-2)^2 - 4(1)(5) = -16 < 0$.

HENCE, $x^2 - 2x + 5$ CANNOT BE FACTORIZED.

TAKE A TEST POINT, SAY $x = 0$. THEN, $0^2 - 2(0) + 5 = 5 > 0$

SO, $x^2 - 2x + 5 > 0$ FOR ALL $x \in (-\infty, \infty)$

THEREFORE, THE SOLUTION SET IS S = $(-\infty, \infty)$

**B** FOR $-3x^2 + x - 1 \geq 0$

$b^2 - 4ac = (1)^2 - 4(-3)(-1) = 1 - 12 = -11 < 0$

HENCE, $-3x^2 + x - 1$ CANNOT BE FACTORIZED. TAKE A TEST POINT, SAY $x = 0$

$-3(0)^2 + 0 - 1 = -1 < 0$. HENCE, $-3x^2 + x - 1 \geq 0$ IS FALSE.

THEREFORE, S = { }

## Group Work 3.3

**1** SOLVE EACH OF THE FOLLOWING INEQUALITIES USING:

**I** PRODUCT PROPERTIES **II** SIGN CHARTS:

**A** $x^2 - \frac{2}{3}x < 0$ **B** $2x^2 + 5x > 3$

**C** $(x-1)^2 \geq 2x^2 - 2x$ **D** $(2x-1)(x+1) \leq x(x-3) + 4$

**2** WHAT MUST BE THE VALUE(S) OF $k$ SO THAT $(k - 4)x^2 + 2kx - 1 = 0$ HAS

**I** TWO DISTINCT REAL ROOTS? **II** ONE REAL ROOT? **III** NO REAL ROOTS?

**3** A MANUFACTURER DETERMINES THAT THE PROFIT OBTAINED FROM SELLING $x$ UNITS OF A CERTAIN ITEM IN BIRR IS $P(x) = 10x - 0.002x^2$

**A** HOW MANY UNITS MUST BE PRODUCED TO SECURE PROFIT?

**B** IN THE PROCESS OF PRODUCTION, AT HOW MANY UNITS WILL THERE BE NO PROFIT AND NO LOSS?

## Exercise 3.4

**1** SOLVE EACH OF THE FOLLOWING QUADRATIC INEQUALITIES USING SIGN CHARTS

**A** $x(x+5) > 0$ **B** $(x-3)^2 \geq 0$

**C** $(4+x)(4-x) < 0$ **D** $\left(1+\frac{x}{3}\right)(5-x) < 0$

**E** $3 - x - 2x^2 > 0$ **F** $-6x^2 + 2 \leq x$

**G** $2x^2 \geq -3 - 5x$ **H** $4x^2 - x - 8 < 3x^2 - 4x + 2$

**I** $-x^2 + 3x < 4$.

**2** SOLVE EACH OF THE FOLLOWING QUADRATIC INEQUALITIES USING THE PRODUCT PROPERTIES OR SIGN CHARTS:

**A** $x^2 + x - 12 > 0$ **B** $x^2 - 6x + 9 > 0$ **C** $x^2 - 3x - 4 \leq 0$

**D** $5x - x^2 < 6$ **E** $x^2 + 2x < -1$ **F** $x - 1 \leq x^2 + 2$

**3** FOR WHAT VALUE(S) OF $k$ DOES EACH OF THE FOLLOWING QUADRATIC EQUATIONS HAVE

**I** ONE REAL ROOT? **II** TWO DISTINCT REAL ROOTS? **III** NO REAL ROOT?

**A** $(k+2)x^2 - (k+2)x - 1 = 0$

**B** $x^2 + (5-k)x + 9 = 0$

**4** FOR WHAT VALUE (S) OF $k$ IS

**A** $kx^2 + 6x + 1 > 0$ FOR EACH REAL NUMBER $x$?

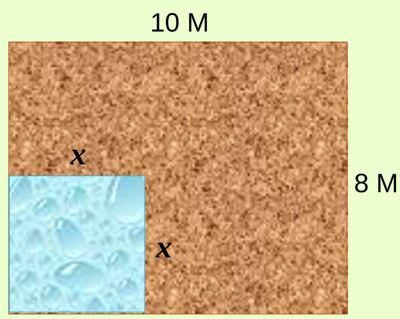
**B** $x^2 - 9x + k < 0$ ONLY FOR $x \in (-2, 11)$?

**5** A ROCKET IS FIRED STRAIGHT UPWARD FROM THE GROUND LEVEL WITH AN INITIAL VELOCITY OF 480 KM/HR. AFTER $t$ SECONDS, ITS DISTANCE ABOVE THE GROUND LEVEL IS GIVEN BY

$480t - 16t^2$.

FOR WHAT TIME INTERVAL IS THE ROCKET MORE THAN 3200 KM ABOVE GROUND LEVEL?

**6** A FARMER HAS 8M BY 10M PLOT OF LAND. HE NEEDS TO CONSTRUCT A WATER RESERVOIR AT ONE CORNER OF THE PLOT WITH EQUAL LENGTH AND WIDTH AS SHOWN BELOW.




FOR WHAT VALUES OF $x$ IS THE AREA OF THE REMAINING PART LESS THAN THE AREA NEEDED FOR THE RESERVOIR?

## 3.3.3 Solving Quadratic Inequalities Graphically

IN ORDER TO USE GRAPHS TO SOLVE QUADRATIC INEQUALITIES, IT IS NECESSARY TO U NATURE OF QUADRATIC FUNCTIONS AND THEIR GRAPHS.

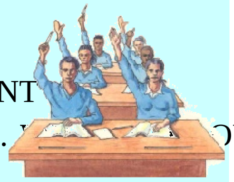
**I** IF $a > 0$, THEN THE GRAPH OF THE QUADRATIC FUNCTION

$f(x) = ax^2 + bx + c$ IS AN ***upward parabola***.

**II** IF $a < 0$, THEN THE GRAPH OF THE QUADRATIC FUNCTION

$f(x) = ax^2 + bx + c$ IS A ***downward parabola***.

### ACTIVITY 3.6



**1** FOR A QUADRATIC FUNCTION $f(x) = ax^2 + bx + c$, FIND THE POINT ... WHICH THE GRAPH TURNS UPWARD OR DOWNWARD. ... OU CALL THIS TURNING POINT?

**2** SKETCH THE GRAPH AND FIND THE TURNING POINT OF:

**A** $f(x) = x^2 - 1$ **B** $f(x) = 4 - x^2$

**3** WHAT IS THE CONDITION FOR THE QUADRATIC FUNCTION $f(x) = ax^2 + bx + c$ TO HAVE A MAXIMUM VALUE? WHEN WILL IT HAVE A MINIMUM VALUE?

**4** WHAT IS THE VALUE OF $x$ AT WHICH THE QUADRATIC FUNCTION $f(x) = ax^2 + bx + c$ ATTAINS ITS MAXIMUM OR MINIMUM VALUE?

THE GRAPH OF A QUADRATIC FUNCTION HAS BOTH ITS ENDS GOING UPWARD OR DEPENDING ON WHETHER $a$ IS POSITIVE OR NEGATIVE. FROM DIFFERENT GRAPHS YOU CAN OE THAT THE GRAPH OF A QUADRATIC FUNCTION

$f(x) = ax^2 + bx + c$

**I** CROSSES THE $x$-AXIS TWICE, IF $b^2 - 4ac > 0$.

**II** TOUCHES THE $x$-AXIS AT A POINT, IF $b^2 - 4ac = 0$.

**III** DOES NOT TOUCH THE $x$-AXIS AT ALL, IF $b^2 - 4ac < 0$.

TO SOLVE A QUADRATIC INEQUALITY GRAPHICALLY, FIND THE VALUES OF $x$ FOR WHICH THE PART OF THE GRAPH OF THE CORRESPONDING QUADRATIC FUNCTION IS ABOVE THE $x$-AXIS, BELOW THE $x$-AXIS OR ON THE $x$-AXIS. CONSIDER THE FOLLOWING EXAMPLES.

**EXAMPLE 5** SOLVE THE QUADRATIC INEQUALITY $x^2 - 3x + 2 < 0$, GRAPHICALLY.

**SOLUTION:** BEGIN BY DRAWING THE GRAPH OF $f(x) = x^2 - 3x + 2$. SOME VALUES FOR $x$ AND $f(x)$ ARE GIVEN IN THE TABLE BELOW AND THE CORRESPONDING GR GIVEN IN FIGURE 3.28. COMPLETE THE TABLE FIRST.

| $x$ | −3 | −2 | −1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | | 12 | | 2 | | 0 | |

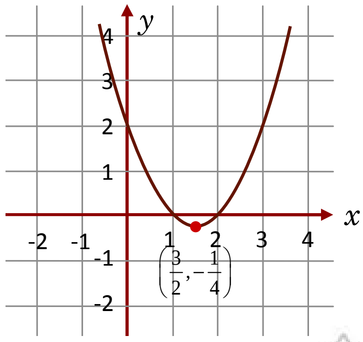



Figure 3.28 *Graph of* $f(x) = x^2 - 3x + 2$

FROM THE GRAPH, $f(x) = 0$ WHEN $x = 1$ AND WHEN $x = 2$. ON THE OTHER HAND, $f(x) > 0$ WHEN $x < 1$ AND WHEN $x > 2$ AND $f(x) < 0$ WHEN $x$ LIES BETWEEN 1 AND 2.

THIS INEQUALITY COULD BE TESTED BY SETTING $x = \frac{3}{2}$, GIVING $f\left(\frac{3}{2}\right) = -\frac{1}{4}$. SO $f\left(\frac{3}{2}\right) < 0$.

IT FOLLOWS THAT THE SOLUTION SET OF $x^2 - 3x + 2 < 0$ CONSISTS OF ALL REAL NUMBERS GREATER THAN 1 AND LESS THAN 2. THAT IS, $S = \{x: 1 < x < 2\} = (1, 2)$.

**EXAMPLE 6** SOLVE THE INEQUALITY $x^2 + 4x + 5 > 0$, GRAPHICALLY.

**SOLUTION:** MAKE A TABLE OF VALUES AND COMPLETE THE TABLE FOR SOME SELECTED VALUES OF $x$ AND $f(x)$ AS IN THE TABLE BELOW AND SKETCH THE CORRESPONDING GRAPH.

| $x$ | −3 | −2 | −1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 2 | | 2 | | 10 | | |

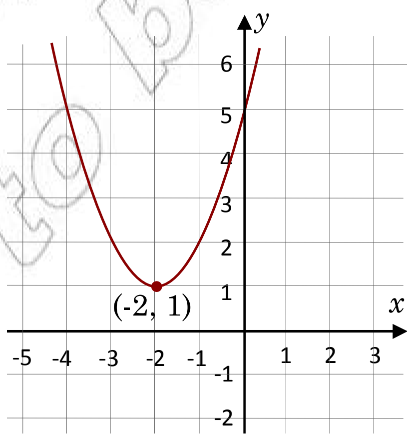



Figure 3.29 Graph of $f(x) = x^2 + 4x + 5$

AS SHOWN IN FIGURE 3.29 ABOVE, THE GRAPH OF $f(x) = x^2 + 4x + 5$ DOES NOT CROSS THE $x$-AXIS BUT LIES ABOVE THE $x$-AXIS. THUS, THE SOLUTION SET OF THIS INEQUALITY CONSISTS OF ALL REAL NUMBERS. SO, S.S = $(-\infty, \infty)$.

NOTE THAT, IF YOU USE THE PROCESS OF COMPLETING THE SQUARE, YOU OBTAIN

$$x^2 + 4x + 5 > 0 \Rightarrow x^2 + 4x > -5$$

$$x^2 + 4x + 4 > -5 + 4$$

$$(x + 2)^2 > -1$$

SINCE THE SQUARE OF ANY REAL NUMBERS IS NON-NEGATIVE, $(x + 2)^2 > -1$ IS TRUE FOR ALL REAL NUMBERS $x$.

BASED ON THE ABOVE INFORMATION, COULD YOU SHOW THAT THE SOLUTION SET OF THE INEQUALITY $x^2 + 4x + 5 < 0$ IS THE EMPTY SET? WHY?

**EXAMPLE 7** SOLVE THE INEQUALITY $-x^2 + 2x + 3 < 0$, GRAPHICALLY.

**SOLUTION:** MAKE A TABLE OF SELECTED VALUES FOR $f(x)$. THE GRAPH PASSES THROUGH (0, 3) AND (–1, 0) AS SHOWN IN FIGURE 3.30.

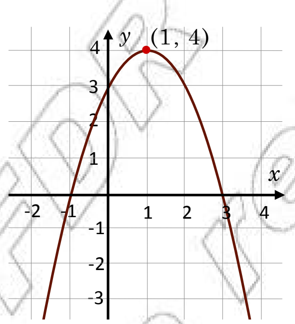



Figure 3.30 Graph of $f(x) = -x^2 + 2x + 3$

THE GRAPH OF $f(x) = 2x - x^2 + 3$ CROSSES THE $x$-AXIS AT $x = -1$ AND $x = 3$. SO, THE SOLUTION SET OF THIS INEQUALITY IS

S.S = $\{x \mid x < -1$ OR $x > 3\}$.

IF THE QUADRATIC EQUATION $ax^2 + bx + c = 0$, $a \neq 0$ HAS DISCRIMINANT $b^2 - 4ac < 0$, THEN THE EQUATION HAS NO REAL ROOTS. MOREOVER,

**i** THE SOLUTION SET OF $ax^2 + bx + c \geq 0$ IS THE SET OF ALL REAL NUMBERS IF $a > 0$ AND IS EMPTY SET IF $a < 0$.

**ii** THE SOLUTION SET OF $ax^2 + bx + c \leq 0$ IS THE SET OF ALL REAL NUMBERS IF $a < 0$ AND IS EMPTY SET IF $a > 0$.

## Exercise 3.5

1. SOLVE EACH OF THE FOLLOWING QUADRATIC INEQUALITIES, GRAPHICALLY:

   **A** $x^2 + 6x + 5 \geq 0$ **B** $x^2 + 6x + 5 < 0$

   **C** $x^2 + 8x + 16 < 0$ **D** $x^2 + 2x + 3 \geq 0$

   **E** $3x - x^2 + 2 < 0$ **F** $4x^2 - x \leq 3x^2 + 2$

   **G** $x(x - 2) < 0$ **H** $(x + 1)(x - 2) > 0$

   **I** $3x^2 + 4x + 1 > 0$ **J** $x^2 + 3x + 3 < 0$

   **K** $3x^2 + 22x + 35 \geq 0$ **L** $6x^2 + 1 \geq 5x.$

2. SUPPOSE THE SOLUTION SET OF $x^2 + kx + 2k > 0$ CONSISTS OF THE SET OF ALL REAL NUMBERS. FIND ALL POSSIBLE VALUES OF $k$

## Key Terms

| | | |
|---|---|---|
| ***absolute value*** | ***linear inequality*** | ***quadratic equation*** |
| ***closed intervals*** | ***open downward*** | ***quadratic function*** |
| ***complete listing*** | ***open intervals*** | ***quadratic inequality*** |
| ***discriminant*** | ***open upward*** | ***sign chart*** |
| ***infinity*** | ***partial listing*** | ***solution set*** |
| ***linear equation*** | ***product property*** | |

## Summary

1. THE OPEN INTERVAL $(a, b)$ WITH END-POINTS $a$ AND $b$ IS THE SET OF ALL REAL NUMBERS $x$ SUCH THAT $a < x < b$.
2. THE CLOSED INTERVAL $[a, b]$ WITH END-POINTS $a$ AND $b$ IS THE SET OF ALL REAL NUMBERS $x$ SUCH THAT $a \leq x \leq b$.
3. THE HALF-OPEN INTERVAL OR HALF-CLOSED INTERVAL $[a, b)$ WITH END-POINTS $a$ AND $b$ IS THE SET OF ALL REAL NUMBERS $x$ SUCH THAT $a \leq x < b$.
4. IF $x$ IS A REAL NUMBER, THEN THE ABSOLUTE VALUE $|x|$ IS DEFINED BY

$$|x| = \begin{cases} x, & \text{IF } x \geq 0 \\ -x, & \text{IF } x < 0 \end{cases}$$

**5** FOR ANY POSITIVE REAL NUMBER $a$, THE SOLUTION SET OF:

**I** THE EQUATION $|x| = a$ IS $x = a$ OR $x = -a$;

**II** THE INEQUALITY $|x| < a$ IS $-a < x < a$ AND

**III** THE INEQUALITY $|x| > a$ IS $x < -a$ OR $x > a$.

**6** WHEN TWO OR MORE LINEAR EQUATIONS INVOLVE THE SAME VARIABLES, THEY ARE CALLED A ***system of linear equations***.

**7** AN INEQUALITY THAT CAN BE REDUCED TO EITHER $ax^2 + bx + c \le 0$, $ax^2 + bx + c < 0$, $ax^2 + bx + c \ge 0$ OR $ax^2 + bx + c > 0$, WHERE $a$, $b$ AND $c$ ARE CONSTANTS AND $a \neq 0$, IS CALLED A ***quadratic inequality***.

**8** GIVEN ANY QUADRATIC EQUATION $ax^2 + bx + c = 0$,

**I** IF $b^2 - 4ac > 0$, IT HAS TWO DISTINCT REAL ROOTS.

**II** IF $b^2 - 4ac = 0$, IT HAS ONLY ONE REAL ROOT.

**III** IF $b^2 - 4ac < 0$, IT HAS NO REAL ROOT.

**9** WHEN THE DISCRIMINANT $b^2 - 4ac < 0$, THEN

**I** THE SOLUTION SET OF $ax^2 + bx + c > 0$ IS THE SET OF ALL REAL NUMBERS, IF $a > 0$ AND EMPTY SET IF $a < 0$.

**II** THE SOLUTION SET OF $ax^2 + bx + c < 0$ IS THE SET OF ALL REAL NUMBERS, IF $a < 0$ AND EMPTY SET IF $a > 0$.

**10** PRODUCT PROPERTY:

**I** $mn > 0$, IF AND ONLY IF $m > 0$ AND $n > 0$ OR $m < 0$ AND $n < 0$.

**II** $mn < 0$, IF AND ONLY IF $m > 0$ AND $n < 0$ OR $m < 0$ AND $n > 0$.

## Review Exercises on Unit 3

**1** SOLVE EACH OF THE FOLLOWING INEQUALITIES USING THE PRODUCT PRINCIPLE.

**A** $(x + 1)(x - 3) < 0$

**B** $\left(\frac{2}{3}x + 3\right)(x - 1) < 0$

**C** $(x - \sqrt{3})(x + \sqrt{2}) > 0$

**D** $x^2 > x$

**E** $x^2 + 5x + 4 \ge 0$

**F** $(x - 2)^2 \le 2 - x$

**G** $1 - 2x \ge (1 + x)^2$

**H** $3x^2 - 6x + 5 < x^2 - 2x + 3$.

**2** SOLVE EACH OF THE FOLLOWING INEQUALITIES USING SIGN CHARTS:

**A** $(1 - x)(5 - x) > 0$ **B** $x^2 \leq 9$ **C** $(x + 2)^2 < 25$

**D** $1 - x \geq 2x^2$ **E** $6t^2 + 1 < 5t$ **F** $2t^2 + 3t \leq 5$.

**3** SOLVE EACH OF THE FOLLOWING INEQUALITIES GRAPHICALLY:

**A** $x^2 - x + 1 > 0$ **B** $x^2 > x + 6$ **C** $x^2 - 4x - 1 > 0$

**D** $x^2 + 25 \geq 10x$ **E** $x^2 + 32 \geq 12x + 6$ **F** $x(6x - 13) > -6$

**G** $x(10 - 3x) < 8$ **H** $(x - 3)^2 \leq 1$

**4** SOLVE EACH OF THE FOLLOWING QUADRATIC INEQUALITIES USING ANY CONVENIENT METHOD:

**A** $2x^2 < x + 2$ **B** $-2x^2 + 6x + 15 \leq 0$

**C** $\frac{1}{2}x^2 + \frac{25}{2} \geq 5x$ **D** $6x^2 - x + 3 < 5x^2 + 5x - 5$

**E** $x(10x + 19) \leq 15$ **F** $(x + 2)^2 > (3x + 1)^2$.

**5** WHAT MUST THE VALUE(S) OF $k$ BE SO THAT:

**A** $kx^2 - 10x - 5 \leq 0$ FOR ALL $x$?

**B** $2x^2 + (k - 3)x + k - 5 = 0$ HAS ONE REAL ROOT? TWO REAL ROOTS? NO REAL ROOT?

**6** THE SUM OF A NON-NEGATIVE NUMBER AND ITS SQUARE IS LESS THAN 12. WHAT COULD THE NUMBER BE?

**7** THE SUM OF A NUMBER $x$ AND TWICE ANOTHER IS 20. IF THE PRODUCT OF THESE NUMBERS IS NOT MORE THAN 48, WHAT ARE ALL POSSIBLE VALUES OF $x$?

**8** THE PROFIT OF A CERTAIN COMPANY IS GIVEN BY $Y(x) = 10,000 + 350x - \frac{1}{2}x^2$

WHERE $x$ IS THE AMOUNT (BIRR IN TENS) SPENT ON ADVERTISING. WHAT AMOUNT GIVES A PROFIT OF MORE THAN BIRR 40,000?